Properties of the Systems

1 System Definition

A system is a device which acts on the input signal $x(t)$ to give an output $y(t)$. It is defined for CT and DT cases respectively as follows

$$x(t) \xrightarrow{\mathbb{H}} y(t) \tag{1}$$

$$x[n] \xrightarrow{\mathbb{H}} y[n] \tag{2}$$

In analysis of systems it is desirable for classifying them according to the properties they satisfy. There are six important such properties which we are studied below.

1.1 Linearity

For the system defined as in (1) let the inputs to the system be $x_1(t)$ and $x_2(t)$ for which the corresponding outputs be $y_1(t)$ and $y_2(t)$ respectively. That is,

$$x_1(t) \xrightarrow{\mathbb{H}} y_1(t) \text{ and } x_2(t) \xrightarrow{\mathbb{H}} y_2(t) \tag{3}$$

If the system satisfies the properties of homogeneity and additivity, defined as

$$\begin{aligned} ax(t) &\xrightarrow{\mathbb{H}} ay(t) && \text{(Homogeneity)} \\ x_1(t) + x_2(t) &\xrightarrow{\mathbb{H}} y_1(t) + y_2(t) && \text{(Additivity)} \end{aligned}$$

we say that the system is linear. By combining these two properties we get linearity property. Formally, for the given input and output relation (3), the system is linear if and only if

$$ax_1(t) + bx_2(t) \xrightarrow{\mathbb{H}} ay_1(t) + by_2(t) \tag{4}$$

for arbitrary scalars a and b.

Similarly a discrete-time (DT) system with

$$x_1[n] \xrightarrow{\mathbb{H}} y_1[n] \text{ and } x_2[n] \xrightarrow{\mathbb{H}} y_2[n] \tag{5}$$

is linear if and only if

$$ax_1[n] + bx_2[n] \xrightarrow{\mathbb{H}} ay_1[n] + by_2[n] \tag{6}$$

Example 1: For the input output relationships given below check whether the systems are linear.

$$(\mathbf{a}) \quad y(t) = x^2(t) \quad (\mathbf{b}) \quad y(t) = x(t^2) \quad (\mathbf{c}) \quad y(t) = x(t)u(t)$$

Solution:

(**a**) For this system,

$$x_1(t) \xrightarrow{\mathbb{H}} y_1(t) = x_1^2(t)$$
$$x_2(t) \xrightarrow{\mathbb{H}} y_1(t) = x_2^2(t)$$

Now,

$$x_3(t) = ax_1(t) + bx_2(t) \xrightarrow{\mathbb{H}} y_3(t) = x_3^2(t) = (ax_1(t) + bx_2(t))^2$$

Similarly,

$$ay_1(t) + by_2(t) = ax_1^2(t) + bx_2^2(t)$$

As $y_3(t) \neq ay_1(t) + by_2(t)$, the given system is nonlinear.

(**b**) For this system,

$$x_1(t) \xrightarrow{\mathbb{H}} y_1(t) = x_1(t^2)$$
$$x_2(t) \xrightarrow{\mathbb{H}} y_1(t) = x_2(t^2)$$

Now,

$$x_3(t) = ax_1(t) + bx_2(t) \xrightarrow{\mathbb{H}} y_3(t) = x_3(t^2) = ax_1(t^2) + bx_2(t^2)$$

Similarly,

$$ay_1(t) + by_2(t) = ax_1(t^2) + bx_2(t^2)$$

As $y_3(t) = ay_1(t) + by_2(t)$, the given system is linear.

(**c**) For this system,

$$x_1(t) \xrightarrow{\mathbb{H}} y_1(t) = x_1(t)u(t)$$
$$x_2(t) \xrightarrow{\mathbb{H}} y_1(t) = x_2(t)u(t)$$

Now,

$$x_3(t) = ax_1(t) + bx_2(t) \xrightarrow{\mathbb{H}} y_3(t) = x_3(t)u(t) = (ax_1(t) + bx_2(t))u(t) = ax_1(t)u(t) + ax_2(t)u(t)$$

Similarly,

$$ay_1(t) + by_2(t) = ax_1(t)u(t) + bx_2(t)u(t)$$

As $y_3(t) = ay_1(t) + by_2(t)$, the given system is linear.

Example 2: For the input output relationships of DT systems given below check whether the systems are linear.

$$(\mathbf{a}) \quad y[n] = x[2^n] \quad (\mathbf{b}) \quad y[n] = (n+1)x(n-1) \quad (\mathbf{c}) \quad y[n] = \sin(x[n])$$

Solution:

(**a**) For this system,

$$x_1[n] \xrightarrow{\mathbb{H}} y_1[n] = x_1[2^n]$$
$$x_2[n] \xrightarrow{\mathbb{H}} y_1[n] = x_2[2^n]$$

Now,

$$x_3[n] = ax_1[n] + bx_2[n] \xrightarrow{\mathbb{H}} y_3[n] = x_3[2^n] = ax_1[2^n] + bx_2[2^n]$$

Similarly,

$$ay_1[n] + by_2[n] = ax_1[2^n] + bx_2[2^n]$$

As $y_3[n] \neq ay_1[n] + by_2[n]$, the given system is linear.

(**b**) For this system,

$$x_1[n] \xrightarrow{\mathbb{H}} y_1[n] = (n+1)x_1[n-1]$$
$$x_2[n] \xrightarrow{\mathbb{H}} y_1[n] = (n+1)x_2[n-1]$$

Now,

$$x_3[n] = ax_1[n] + bx_2[n] \xrightarrow{\mathbb{H}} y_3[n] = (n+1)x_3[n-1] = (n+1)(ax_1[n-1] + bx_2[n-1])$$

Similarly,

$$ay_1[n] + by_2[n] = a(n+1)x_1[n-1] + b(n+1)x_2[n-1]$$

As $y_3[n] = ay_1[n] + by_2[n]$, the given system is linear.

(**c**) For this system,

$$x_1[n] \xrightarrow{\mathbb{H}} y_1(t) = \sin(x_1[n])$$
$$x_2[n] \xrightarrow{\mathbb{H}} y_1(t) = \sin(x_2[n])$$

Now,

$$x_3[n] = ax_1[n] + bx_2[n] \xrightarrow{\mathbb{H}} y_3[n] = \sin(x_3[n]) = \sin(ax_1[n] + bx_2[n])$$

Similarly,

$$ay_1[n] + by_2[n] = a\sin(x_1[n]) + b\sin(x_1[n])$$

As $y_3[n] \neq ay_1[n] + by_2[n]$, the given system is nonlinear.

1.2 Time Invariance

A system is said to be time-invariant if any shift of the input signal results in same amount of the shift in the output. That is, the CT system(1), is time-invariant if and only if

$$x(t-t_0) \xrightarrow{\mathbb{H}} y(t-t_0) \tag{7}$$

Similarly a DT system $x[n] \xrightarrow{\mathbb{H}} y[n]$ is time-invariant if and only if

$$x[n-n_0] \xrightarrow{\mathbb{H}} y[n-n_0] \tag{8}$$

Example 3: For the CT systems given in **Example 1**, check for the time-invariance property.

Solution:

(**a**) For this system,

$$x_1(t) = x(t-t_0) \xrightarrow{\mathbb{H}} y_1(t) = x_1^2(t) = x^2(t-t_0)$$

Similarly,

$$y(t-t_0) = x^2(t-t_0)$$

As $y(t-t_0) = y_1(t)$, the given system is time-invariant.

(**b**) For this system,

$$x_1(t) = x(t-t_0) \xrightarrow{\mathbb{H}} y_1(t) = x_1(t^2) = x(t^2-t_0)$$

Similarly,

$$y(t-t_0) = x((t-t_0)^2)$$

As $y(t-t_0) \neq y_1(t)$, the given system is time-variant.

(**c**) For this system,

$$x_1(t) = x(t-t_0) \xrightarrow{\mathbb{H}} y_1(t) = x_1(t)u(t) = x(t-t_0)u(t)$$

Similarly,

$$y(t-t_0) = x(t-t_0)u(t-t_0)$$

As $y(t-t_0) \neq y_1(t)$, the given system is time-variant.

Example 4: For the DT systems given in **Example 2**, check for the time-invariance property.

Solution:

(**a**) For this system,

$$x_1[n] = x[n-n_0] \xrightarrow{\mathbb{H}} y_1[n] = x_1[2^n] = x[2^n-n_0]$$

Similarly,

$$y[n-n_0] = x[2^{n-n_0}]$$

As $y[n-n_0] \neq y_1[n]$, the given system is time-variant.

(**b**) For this system,

$$x_1[n] = x[n-n_0] \xrightarrow{\mathbb{H}} y_1[n] = (n+1)x_1[n-1] = (n+1)x[n-n_0-1]$$

Similarly,

$$y[n-n_0] = (n-n_0+1)x[n-n_0-1]$$

As $y[n-n_0] \neq y_1[n]$, the given system is time-variant.

(**c**) For this system,

$$x_1[n] = x[n-n_0] \xrightarrow{\mathbb{H}} y_1[n] = \sin(x_1[n]) = \sin(x[n-n_0])$$

Similarly,

$$y[n-n_0] = \sin(x[n-n_0])$$

As $y[n-n_0] = y_1[n]$, the given system is time-invariant.

1.3 Causality

A CT system is causal if present value of the output $y(t)$ depends on past values of the input and/or present value of the input $x(t)$.

A DT system is causal if present value of the output $y[n]$ depends on past values of the input and/or present value of the input $x[n]$.

Example 5: For the CT systems given in **Example 1**, check for the causality property.

Solution:

(**a**) For this system, we write

$$y(t) = [x(t)]^2$$

As present value of the output depends only on present value of the input, it is a causal system.

(**b**) For this system, for example, $y(2) = x(4)$, i.e., present value of the output depends on future value of the input, it is a noncausal system.

(**c**) For this system, present value of the output depends on present value of the input, it is a causal system.

Example 6: For the DT systems given in **Example 2**, check for the causality property.

Solution:

(**a**) For this system, for example, $y[2] = x[4]$, i.e., present value of the output depends on future value of the input, it is a noncausal system.

(**b**) For this system, present value of the output depends only on past value of the input, it is a causal system.

(**c**) For this system, present value of the output depends on present value of the input, it is a causal system.

1.4 Memorylessness

A CT system is memoryless if present value of the output $y(t)$ depends only on present value of the input $x(t)$.

A DT system is memoryless if present value of the output $y[n]$ depends only on present value of the input $x[n]$.

Example 7: For the CT systems given in **Example 1**, check for the memorylessness property.

Solution:

(**a**) For this system, present value of the output depends only on present value of the input, it is a memoryless system.

(**b**) For this system, for example, $y(2) = x(4)$, i.e., present value of the output depends on future value of the input, it is not a memoryless system.

(**c**) For this system, present value of the output depends only on present value of the input, it is a memoryless system.

Example 8: For the DT systems given in **Example 2**, check for the memorylessness property.

Solution:

(**a**) For this system, for example, $y[2] = x[4]$, i.e., present value of the output depends on future value of the input, it is not a memoryless system.

(**b**) For this system, present value of the output depends on past value of the input, it is a not a memoryless system.

(**c**) For this system, present value of the output depends on present value of the input, it is a memoryless system.

1.5 Stability

Any CT signal $x(t)$ or a DT signal $x[n]$, is said to be bounded if

$$\begin{aligned} |x(t)| \leq B_x < \infty \qquad & \forall t \in \mathbb{R} \qquad \text{CT case} \\ |x[n]| \leq B_x < \infty \qquad & \forall n \in \mathbb{Z} \qquad \text{DT case} \end{aligned}$$

If we apply this bounded input to any system and if it produces a bounded output then we say that the system is stable. That is,

$$\begin{aligned} |y(t)| \leq B_y < \infty \qquad & \forall t \in \mathbb{R} \qquad \text{CT case} \\ |y[n]| \leq B_y < \infty \qquad & \forall n \in \mathbb{Z} \qquad \text{DT case} \end{aligned}$$

Example 9: For the CT systems given in **Example 1**, check for the stability property.

Solution:

(**a**) For this system, if we apply a bounded input, the output is also bounded as, if $|x(t)|$ is bounded by B_x, the output $|y(t)|$ will be bounded by $B_y = B_x^2 < \infty$.

(**b**) For this system, if we apply a bounded input, the output is also bounded as, if $|x(t)|$ is bounded by B_x, the output $|y(t)|$ will be bounded by $B_y = B_x < \infty$.

(**c**) For this system, if we apply a bounded input, the output is also bounded as if $|x(t)|$ is bounded by B_x, the output $|y(t)|$ will be bounded by $B_y = B_x < \infty$.

Example 10: For the DT systems given in **Example 2**, check for the stability property.

Solution:

(**a**) For this system, if we apply a bounded input, the output is also bounded as, if $|x[n]|$ is bounded by B_x, the output $|y[n]|$ will be bounded by $B_y = B_x < \infty$.

(**b**) For this system, if we apply a bounded input, the output is not bounded as, if $|x[n]|$ is bounded by B_x, the output $|y[n]|$ will not be bounded as for $n = \infty$, $y[n] = \infty$ which is unbounded.

(**c**) For this system, if we apply a bounded input, the output is also bounded as if $|x[n]|$ is bounded by B_x, the output $|y[n]|$ will be bounded by $-1 \leq B_y \leq 1$.

1.6 Invertibility

A CT system is said to be invertible if and only if we can uniquely determine the input $x(t)$ from the output $y(t)$.

Example 11: For the CT systems given in **Example 1**, check for the invertibility property. Similar definition holds for DT case also.

Solution:

(**a**) For this system, as $\pm x(t)$ produces the same output $y(t)$, we cannot uniquely determine the input $x(t)$ from $y(t)$. Thus this system is not invertible.

(**b**) For this system, we can uniquely determine the input $x(t)$ from $y(t)$ as $y(\pm\sqrt{t})$. Thus this system is invertible.

(**c**) For this system, we cannot uniquely determine the input $x(t)$ from $y(t)$ as for $t < 0$ $x(t)$ is not defined. Thus this system is not invertible.

Example 12: For the DT systems given in **Example 2**, check for the invertibility property.

Solution:

(**a**) For this system, though $y[\log_2 n]$ gives $x[n]$ for $n > 0$, for $n \leq 0$, $\log_2 n$ is undefined. Thus this system is not invertible.

(**b**) For this system, though $y[n+1]/(n+2)$ gives $x[n]$ for $n \neq -2$, for $n = -2$, $x[n]$ is undefined. Thus this system is not invertible.

(**c**) For this system, though $\sin^{-1} y[n]$ gives $x[n]$ but $x[n] \pm 2n\pi$ also gives same $y[n]$. Thus this system is not invertible.